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THE FIRST EXOTIC CLASS OF A MANIFOLD

BY
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Let BF be the classifying space for stable oriented spherical fibrations. Gitler and Stasheff have defined a cohomology class e_1 in $H^{pr-1}(BF; Z_p)$, where here (and throughout this paper) p is an odd prime and $r=2(p-1)$. For a discussion of the nature and significance of this class, see the introductions to [2], [4], [7].

Suppose P is a $(pr-1)$-dimensional oriented Poincaré complex. Let ν be the stable normal spherical fibration of P, i.e., the unique stable spherical fibration with reducible Thom complex [6]. Define $e_1(P)$ to be $e_1(\nu)$. Similarly, define $q_i(P)$ to be $q_i(\nu)$, where q_i is the ith Wu class. It is clear that $e_1(P)$ depends only on the homotopy type of P. We wish, however, to express $e_1(P)$ in terms of an explicit invariant of the homotopy type of P.

In fact, we will construct a certain nonstable secondary cohomology operation Ω, mapping cohomology classes of dimension r into classes of dimension $pr-1$ (Z_p coefficients), such that

THEOREM 1. *$\Omega(q_1P)$ is defined with zero indeterminacy, and $\Omega(q_1P)=e_1(P)$.*

REMARK. In the course of the proof of Theorem 1, we will give a construction of the class e_1 which is rather different from that given by Gitler and Stasheff.

Theorem 1 allows us to compute $e_1(P)$—at least in principle, and often in practice. For example,

THEOREM 2. *There is a Poincaré complex P of the homotopy type of*

$$(S^r \vee S^{(p-1)r-1}) \cup e^{pr-1}$$

such that $e_1(P)\neq 0$.

In another paper these results will be applied to the study of the existence of differential structures on manifolds of dimension $pr-1$. Indeed, let Γ_i be the group of exotic i-dimensional spheres, and let Γ'_i be the quotient of Γ_i by the subgroup of spheres bounding parallelizable manifolds. According to [3], p-torsion first appears in the groups Γ'_i precisely when $i=pr-2$; in fact, if ${}^pG$ denotes the p-primary component of G, then ${}^p\Gamma'_{pr-2}=Z_p$. Moreover, $\Gamma_i=\Gamma'_i$ for i even, so ${}^p\Gamma_{pr-2}=Z_p$. Now let M be a differential manifold (all manifolds are compact and oriented) of

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dimension $pr-1$, whose boundary ∂M is an exotic sphere. Let $[\partial M]_p \in {}^p\Gamma_{pr-2}$ be the p-primary component of this exotic sphere. Also, let M^* be the closed topological manifold $M \cup \text{Cone}(\partial M)$.

THEOREM 3. $[\partial M]_p=0$ *if and only if* $\Omega(q_1 M^*)=0$.

Thus $\Omega(q_1 M^*)$ measures the p-primary component of the obstruction to smoothing M^*. Using this result, one can construct a manifold M whose boundary is a generator Σ_p of ${}^p\Gamma_{pr-2}$.

THEOREM 4. Σ_p *bounds a manifold of the homotopy type of* $S^r \vee S^{(p-1)r-1}$.

We postpone proofs of Theorems 3 and 4 to a later paper, where results on higher-dimensional exotic spheres not bounding parallelizable manifolds will also be given.

REMARK. A proof of Theorem 4 for the case $p=3$ was given in [1]. A similar proof does not work when $p \geqq 5$ because the appropriate element of $\pi_{pr-2}(S^r)$ is in the image of the J-homomorphism only when $p=3$. In particular, Σ_p bounds a "plumbed" manifold only when $p=3$.

The proof of Theorem 1 uses recent techniques of E. Thomas. I am very grateful to Professor Thomas for teaching me about his methods.

1. **Definition of the operation Ω.** Let $q_i \in H^{ir}(BSO; Z_p)$ be the ith Wu class; let P^i be the ith Steenrod power (for the prime p).

LEMMA 1.1. *If $j<p$, then*

$$q_j \doteq b_j P^{j-1}(q_1) + R_j(q_1, P^1(q_1), \ldots, P^{j-2}(q_1)),$$

where $b_j \in Z_p$ is nonzero, and R_j is a polynomial in $j-1$ variables.

Proof. If $j=1$, clear. Assume the lemma for all $t<j$; let U be the universal Thom class. Then

$$\begin{aligned} q_j \cdot U = P^j(U) &= b_j P^{j-1} P^1(U), \quad \text{by an Adem relation} \\ &= b_j P^{j-1}(q_1 \cdot U) \\ &= b_j \Big[P^{j-1}(q_1)\cdot U + \sum_t P^{j-1-t}(q_1)\cdot P^t(U) \Big], \end{aligned}$$

where the summation runs from 1 to $j-1$,

$$= b_j \Big[P^{j-1}(q_1)\cdot U + \sum_t P^{j-1-t}(q_1)\cdot q_t \cdot U \Big].$$

Therefore,

$$q_j = b_j \Big[P^{j-1}(q_1) + \sum_t P^{j-1-t}(q_1)\cdot q_t \Big].$$

Since, by assumption, q_t is a polynomial in $q_1, \ldots, P^{t-1}(q_1)$, the lemma follows by induction.

To define the operation Ω, we will display its universal example. Let K_i ($i>0$) be an Eilenberg-MacLane space $K(Z_p, i)$ and let u_i be the fundamental class in $H^i(K_i)$ (Z_p coefficients always).

We define a class $v \in H^{(p-1)r}(K_r)$ by the formula

$$v = b_{p-1}P^{p-2}(u_r)+R_{p-1}(u_r, P^1(u_r), \ldots, P^{p-3}(u_r)),$$

where b_{p-1} and R_{p-1} are as in Lemma 1.1.

We have the diagram

$$\begin{array}{ccc} K_{(p-1)r-1} & \xrightarrow{i} & E & & \\ & & \downarrow{\scriptstyle \pi} & & \\ & & K_r & \xrightarrow{v} & K_{(p-1)r} \end{array}$$

where π is the fibration induced by v from the path-loop fibration on $K_{(p-1)r}$, and i is the inclusion of the fiber. (We refer to such a diagram as the principal fibration determined by v.)

Associated to this fibration we have the Thomas exact sequence ([10, p. 187]; see also, [5], [11])

$$(*) \qquad H^j(K_r) \xrightarrow{\pi} H^j(E) \xrightarrow{\mu} H^j(K_{(p-1)r-1}\times E, E) \xrightarrow{\tau} H^{j+1}(K_r),$$

valid for all $j<2(p-1)r-2$.

Define $\alpha \in H^{pr-1}(K_{(p-1)r-1}\times E, E)$ by

$$\alpha = P^1u \otimes 1+u \otimes \pi^*(u_r),$$

where $u=u_{(p-1)r-1}$.

LEMMA 1.2. $\tau(\alpha)=0$.

Proof. Using properties of τ [10, p. 188], we have

$$\tau(\alpha) = P^1(\bar{\tau}(u))+\bar{\tau}(u) \cup u_r = P^1v+v \cup u_r,$$

where $\bar{\tau}$ is the usual transgression.

Let f: $BSO \to K_r$ satisfy $f^*(u_r)=q_1$. By [2, Lemma 3.3], kernel f^* is generated, in dimensions $\leqq pr$, by those elements in the Cartan basis containing a Bockstein. Therefore, it suffices to show $f^*(\tau(\alpha))=0$. But note that $f^*(v)=q_{p-1}$. Thus

$$\begin{aligned} f^*(\tau(\alpha)) = f^*(P^1v+v \cup u_r) &= P^1(q_{p-1})+q_{p-1} \cup q_1 \\ &= 0 \quad \text{(by a Wu formula)} \end{aligned}$$

which proves the lemma.

Since $\tau(\alpha)=0$, there is, by exactness of (*), an $\omega \in H^{pr-1}(E)$ with $\mu(\omega)=\alpha$. We take ω to be the universal representative for the operation Ω. Thus Ω is defined on those classes $x \in H^r(X)$ for which $v(x)=0$; the indeterminacy subgroup of $\Omega(x)$ is the image of the homomorphism $H^{(p-1)r-1}(X) \to H^{pr-1}(X)$ given by (see [10], [5]) $c \to P^1c+c \cup x$.

2. **A result of Thomas.** We quote a result which we will need in the proof of Theorem 1. Suppose we have a principal fibration

$$\begin{array}{ccc} K_{jr-1} & \xrightarrow{i} & E \\ & & \downarrow \pi \\ & & B \xrightarrow{c} K_{jr}. \end{array}$$

Let $k \in H^t(E)$, where $t<2jr-2$, and let $i^*(k)=\alpha(u_{jr-1})$ for some α in A_p, the Steenrod algebra. Assume there is a spherical fibration ξ over B with $q_j(\xi)=c$. Moreover, assume there is an Adem relation $\alpha P^j=0$, and let Φ be an associated secondary operation. Let U_E be the Thom class of $\pi^*(\xi)$. Note that Φ is defined on U_E.

PROPOSITION 2.1. *There is a class $d \in H^t(E)$ such that $i^*(d)=i^*(k)$ and*

$$d\cdot U_E \in \Phi(U_E).$$

Proof. This is a special case of the mod p analogue of Theorem 6.4 of [9]. (Set $B'=$point, $k'+p^*m=d$ in that theorem. Thomas states his result only for vector bundles, but the proof is the same when ξ is a spherical fibration.)

3. **Proof of Theorem 1.** We deduce Theorem 1 from a more general result. Let β be a $(k-1)$-dimensional oriented spherical fibration over a complex X. Let $T(\beta)$ be the Thom complex of β and U_β the Thom class in $H^k(T(\beta))$. Let Φ be the secondary cohomology operation associated to the Adem relation $P^1P^{p-1}=0$. Assume that the Wu class $q_{p-1}(\beta)=0$. Note that $\Phi(U_\beta)$ is then defined.

THEOREM A. *Given β as above, $\Omega(q_1(\beta))$ is defined and*

$$[\Omega(q_1(\beta))-e_1(\beta)]\cdot U_\beta = \Phi(U_\beta).$$

In particular, these two expressions have the same indeterminacy.

Before proving Theorem A, we give the

Proof of Theorem 1. Let ν be the normal spherical fibration of P. Since $T(\nu)$ is S-dual to P^+ ($=P$ with a disjoint basepoint), $P^{p-1}(U_\nu)=0$ if and only if the homomorphism

$$c(P^{p-1})\colon H^{r-1}(P) \to H^{pr-1}(P)$$

is trivial, where c is the canonical anti-isomorphism of A_p. (Compare [8, Chapter III, Proposition 1.4].) But $c(P^{p-1})$ is a multiple of P^{p-1}, and P^{p-1} is zero on $(r-1)$-dimensional classes. Therefore $P^{p-1}(U_\nu)=0$, i.e., $q_{p-1}(\nu)=0$.

Since $q_{p-1}(\nu)=0$, we may apply Theorem A to ν. Now $T(\nu)$ is reducible (that is, the top cohomology class is spherical), so $\Phi(U_\nu)=0$, with zero indeterminacy. Therefore

$$[\Omega(q_1(\nu))-e_1(\nu)]\cdot U_\nu = 0,$$

so $\Omega(q_1P)=e_1(P)$, with zero indeterminacy, which proves Theorem 1.

Proof of Theorem A. First note that $v(q_1(\beta))=q_{p-1}(\beta)=0$, so, Ω is defined on $q_1(\beta)$.

Also, the indeterminacy of $\Phi(U_\beta)$ consists of all elements of the form $P^1(c\cdot U)$, $c\in H^{(p-1)r-1}(X)$. If $\{P^1(c\cdot U)\}$ denotes the set of all such elements, then

$$\begin{aligned}\{P^1(c\cdot U)\} &= \{[P^1(c)+c\cup q_1(\beta)]\cdot U\}\\ &= [\text{indeterminacy of } \Omega(q_1(\beta))]\cdot U.\end{aligned}$$

Therefore $[\Omega(q_1(\beta))-e_1(\beta)]\cdot U_\beta$ and $\Phi(U_\beta)$ have the same indeterminacy. Thus to prove Theorem A, we need only show they have a common representative.

Let $BF(m)$, m large, be the classifying space for $(m-1)$-dimensional oriented spherical fibrations, and let ξ be the universal $(m-1)$-dimensional fibration.

Consider the principal fibration

$$\text{(3.1)}\qquad \begin{array}{ccc} K_{(p-1)r-1} & \xrightarrow{i_1} & E_1 \\ & & \downarrow{\scriptstyle \pi_1} \\ & & BF(m) \xrightarrow{q_{p-1}} K_{(p-1)r}. \end{array}$$

Since $q_{p-1}=v(q_1)$, we have a commutative diagram

$$\begin{array}{ccccc} K_{(p-1)r-1} & = & K_{(p-1)r-1} & & \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i} & & \\ E_1 & \xrightarrow{g} & E & & \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ BF(m) & \xrightarrow{q_1} & K_r & \xrightarrow{v} & K_{(p-1)r}. \end{array}$$

Let $k=g^*(\omega)$. Then, by definition, $k\in\Omega(\pi_1^*(q_1))$.

Let δ over E_1 be the spherical fibration $\pi_1^*(\xi)$. We claim

$$(*)\qquad (k-e_1(\delta))\cdot U_\delta\in\Phi(U_\delta).$$

Notice that Theorem A follows immediately from (*), for if $\beta\colon X\to BF(m)$ is a spherical fibration with $q_{p-1}(\beta)=0$, then β lifts to $f\colon X\to E_1$. By naturality, $(f^*(k)-e_1(\beta))\cdot U_\beta\in\Phi(U_\beta)$ and $f^*(k)\in\Omega(q_1(\beta))$. Thus $(f^*(k)-e_1(\beta))\cdot U_\beta$ is the required common representative.

To show (*), we apply Proposition 2.1 to the fibration (3.1), with $\alpha=P^1$. We conclude that there is a class $d\in H^{pr-1}(E_1)$ with $i^*(d)=i^*(k)$ and

$$(**)\qquad d\cdot U_\delta\in\Phi(U_\delta).$$

LEMMA 3.2. *$k=d+\pi_1^*(z)$, for a unique $z\in H^{pr-1}(BF(m))$.*

Proof. It is easy to see that $H^i(BF)=0$, $i<r$, and $H^r(BF)=Z_p$, with q_1 as

generator. (For example, see [4].) It follows that any $\alpha \in H^{pr-1}(K_{(p-1)r-1} \times E_1, E_1)$ can be written in the form

$$aP^1(u) \otimes 1 + u \otimes \pi_1^*(b \cdot q_1), \qquad a, b \in Z_p.$$

Let

$$\mu\colon H^{pr-1}(E_1) \to H^{pr-1}(K_{(p-1)r-1} \times E_1, E_1)$$

be the map in the Thomas exact sequence for the fibration (3.1). If $i_1^*(x) = 0$, $x \in H^{pr-1}(E_1)$, then $\mu(x) = u \otimes \pi_1^*(b \cdot q_1)$. Now $0 = \tau\mu(x) = b(q_{p-1} \cup q_1)$. This implies $b = 0$, since $q_{p-1} \cup q_1 \neq 0$. Therefore, $i_1^*(x) = 0$ implies $\mu(x) = 0$. By exactness, $x = \pi_1^*(z)$. Moreover, a glance at the Thomas exact sequence shows that

$$\pi_1^*\colon H^{pr-1}(BF(m)) \to H^{pr-1}(E_1)$$

is injective, so z is unique. Since $i_1^*(k-d) = 0$, the lemma follows.

LEMMA 3.3. *The class z which occurs in Lemma 3.2 is nonzero.*

Proof. Let $h\colon S^{pr-1} \to BF(m)$ be a generator of ${}^p\pi_{pr-1}(BF(m)) = Z_p$. Note that h lifts to $f\colon S^{pr-1} \to E_1$. By Lemma 3.2, $f^*(k) = f^*(d) + h^*(z)$.

Now $f^*(k) \in \Omega(q_1(h)) = 0$, with zero indeterminacy, so $f^*(d) = -h^*(z)$.

By (**), $f^*(d) \cdot U_h \in \Phi(U_h)$.

But the Thom complex $T(h)$ is of the form $S^m \cup_{\beta_1} e^{m+pr-1}$, where β_1 is a generator of ${}^p\pi_{pr-2}$ (see [2]), and Φ acts nontrivially in this complex, i.e., $\Phi(U_h) \neq 0$, with zero indeterminacy. Therefore $z \neq 0$.

We now *define* e_1 to be the class z. Then

$$d = k - \pi_1^*(e_1) = k - e_1(\delta).$$

Together with (**), this shows (*), and proves Theorem A. Alternatively, we may show that z is equal to the class defined in [2], which we will call $\bar{e}_1$. Since $H^{pr-1}(BF)$ is at most Z_p [4], $\bar{e}_1 = bz$, $b \in Z_p$ nonzero. According to [2], if Ψ is the secondary operation corresponding to the relation $P^1(P^1)^{p-1} = 0$, then $h^*(\bar{e}_1) \cdot U_h \in \Psi(U_h)$. Now $(P^1)^{p-1} = (p-1)!\,P^{p-1} = -P^{p-1}$, so $-h^*(\bar{e}_1) \cdot U_h \in \Phi(U_h)$. But $-h^*(z) \cdot U_h \in \Phi(U_h)$. Therefore $\bar{e}_1 = z$.

4. **Proof of Theorem 2.** Theorem 2 follows easily from

PROPOSITION 4.1. *There is a $\beta \in \pi_{pr-2}(S^r)$ such that if $X = S^r \cup_\beta e^{pr-1}$, then $\Omega\colon H^r(X) \to H^{pr-1}(X)$ is nontrivial. (Clearly Ω is defined with zero indeterminacy.)*

Assuming Proposition 4.1, we prove Theorem 2. Recall that

$$\pi_{i+j-1}(S^i \vee S^j) = \pi_{i+j-1}(S^i) \oplus \pi_{i+j-1}(S^j) \oplus Z, \qquad (i,j \geqq 2)$$

where the infinite cyclic factor is generated by the Whitehead product $[e_i, e_j]$ of the inclusions $e_t\colon S^t \to S^i \vee S^j$, $t = i, j$.

If $f \in \pi_{i+j-1}(S^i \vee S^j)$, we write

$$f = f_i \oplus f_j \oplus H_f, \qquad f_t \in \pi_{i+j-1}(S^t), \quad H_f \in Z.$$

If $i \neq j$, it is easy to see that $(S^i \vee S^j) \cup_f e^{i+j}$ is a Poincaré complex if and only if $H_f = \pm 1$. We write $P(f_i, f_j)$ for the Poincaré complex $(S^i \vee S^j) \cup_f e^{i+j}$, where $f = f_i \oplus f_j \oplus 1$. Now let $i=r$, $j=(p-1)r-1$. Let $\alpha \in \pi_{pr-2}(S^{(p-1)r-1})$ be a map such that $P^1: H^{(p-1)r-1}(Y) \to H^{pr-1}(Y)$ is nontrivial, where $Y = S^{(p-1)r-1} \cup_\alpha e^{pr-1}$ (see [8, p. 89]). Let $P = P(\beta, \alpha)$, with β as in Proposition 4.1.

We claim $e_1(P) \neq 0$. By Theorem 1, $\Omega(q_1 P)$ is defined with zero indeterminacy, and $e_1(P) = \Omega(q_1 P)$. But there is a map $c: P \to X = S^r \cup_\beta e^{pr-1}$ such that $c^*: H^t(X) \to H^t(P)$ is an isomorphism for $t \neq (p-1)r-1$ (collapse $S^{(p-1)r-1}$ to a point). By naturality, $\Omega: H^r(P) \to H^{pr-1}(P)$ is an isomorphism. Thus it suffices to show $q_1(P) \neq 0$; this follows from the fact that $P^1: H^{(p-1)r-1}(P) \to H^{pr-1}(P)$ is nontrivial.

This proves Theorem 2, except for the proof of Proposition 4.1, which we now give. The following lemma is certainly known to the dedicated homotopy theorists (for completeness, we give a proof).

LEMMA 4.2. *There exist a complex*

$$L = S^{2r-2} \cup e^{3r-2} \cup \cdots \cup e^{(p-1)r-2}$$

and maps

$$g: L \to S^{r-1}, \qquad h: S^{pr-3} \to L$$

such that

(1) $P^1: H^{ir-2}(L) \to H^{(i+1)r-2}(L)$ *is nontrivial*, $2 \leqq i \leqq p-2$.

(2) *The functional operation* $P^1_g: H^{r-1}(S^{r-1}) \to H^{2r-2}(L)$ *is nontrivial.*

(3) *The functional operation* $P^1_h: H^{(p-1)r-2}(L) \to H^{pr-3}(S^{pr-3})$ *is nontrivial.*

Proof. Suppose inductively that we have constructed a complex

$$L(i-1) = S^{2r-2} \cup e^{3r-2} \cup \cdots \cup e^{(i-1)r-2}, \qquad i \leqq p-1,$$

and a map $g(i-1): L(i-1) \to S^{r-1}$ such that the functional operation $P^1_{g(i-1)}$ is nontrivial.

We define

$$L(i) = \sum [S^{r-1} \cup_{g(i-1)} \text{Cone}\,(L(i-1))],$$

where $\sum$ is the $(r-1)$-fold suspension.

To define $g(i): L(i) \to S^{r-1}$, we use the following lemma.

LEMMA 4.3. *Let (A, B) be a pair of complexes which is the suspension of a pair (A', B'), and let Y be a space which is n-simple for all n. Suppose $f: B \to Y$ is of order p in the group of homotopy classes $[B, Y]$. If, for all i, $H^{i+1}(A, B; \pi_i(Y))$ is finite and has trivial p-primary component, then f extends over A.*

Proof. Left to reader. (Hint: Given a partial extension F of f, the obstruction cohomology class c_F has order t, $t \not\equiv 0(p)$. Take s with $st \equiv 1(p)$. Then $st(F)|B = f$ and $c_{st(F)} = 0$.)

Let $\alpha: S^{2r-2} \to S^{r-1}$ be a map which is of order p in $\pi_{2r-2}(S^{r-1})$; then the functional operation P^1_α is nontrivial [8, p. 90]. Since ${}^p\pi_{jr-3}(S^{r-1})=0$ for $j \leqq p-1$ [12], Lemma 4.3 implies that α extends to a map $g(i): L(i) \to S^{r-1}$.

Thus we may construct $L(p-1)$ and $g(p-1): L(p-1) \to S^{r-1}$. We may also construct the complex $L(p)$ (but not the map $g(p)$). Let

$$\begin{aligned} L &= ((p-1)r-2)\text{-skeleton of } L(p), \\ h: S^{pr-3} \to L &= \text{attaching map of the } (pr-2)\text{-cell of } L(p), \\ g: L \to S^{r-1} &= \text{an extension of } \alpha: S^{2r-2} \to S^{r-1}. \end{aligned}$$

It is easy to see that the L, g, and h we have constructed satisfy conditions (1), (2), and (3).

Now let $\beta' \in \pi_{pr-3}(S^{r-1})$ be the composite gh, and let β be the suspension of β'. Also, let

$$X' = S^{r-1} \cup_{\beta'} e^{pr-2}, \qquad X = \text{suspension of } X' = S^r \cup_\beta e^{pr-1}.$$

To show $\Omega: H^r(X) \to H^{pr-1}(X)$ is nontrivial, it suffices to show

$$\Omega': H^{r-1}(X') \to H^{pr-2}(X')$$

is nontrivial, where Ω' is the secondary operation whose universal example is obtained by applying the loop functor to the universal example for Ω. Thus the universal example for Ω' is

$$\begin{array}{ccc} K_{(p-1)r-2} \longrightarrow E' & & \\ \qquad\qquad\downarrow & & \\ \qquad\qquad K_{r-1} & \xrightarrow{v'} & K_{(p-1)r-1}. \end{array}$$

Since $v = bP^{p-2} + \text{product terms}$, $b \in Z_p$ nonzero, $v' = bP^{p-2}$. (Informally, Ω' is the secondary operation associated to the relation $P^1P^{p-2}=0$ on classes of dimension $r-1$.)

There is a commutative ladder of cofibrations

$$\begin{array}{ccccccc} S^{pr-3} & \xrightarrow{h} & L & \longrightarrow & L \cup_h e^{pr-2} & \longrightarrow & S^{pr-2} \\ \| & & \downarrow g & & \downarrow F & & \| \\ S^{pr-3} & \xrightarrow{\beta'} & S^{r-1} & \longrightarrow & X' & \longrightarrow & S^{pr-2}. \end{array}$$

Using conditions (1) and (2) of Lemma 4.2 (and the fact that $P^{p-2}=b'(P^1)^{p-2}$), we see that the functional operation

$$P_F^{p-2}: H^{r-1}(X') \to H^{(p-1)r-2}(L \cup e^{pr-2})$$

is nontrivial. Using condition (3) of Lemma 4.2, we see that

$$P^1: H^{(p-1)r-2}(L \cup_h e^{pr-2}) \to H^{pr-2}(L \cup_h e^{pr-2})$$

is nontrivial. A Peterson-Stein formula [5] shows that

$$\Omega': H^{r-1}(X') \to H^{pr-2}(X')$$

is nontrivial. This completes the proof of Proposition 4.1.

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